Lecture # 12: Shock Waves and De Laval Nozzle

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Sources/ Further reading:
Anderson, "Fundamentals of Aerodynamics" Part 3 Chs 7 & 8

Subsonic, Transonic, Supersonic and hypersonic Flows

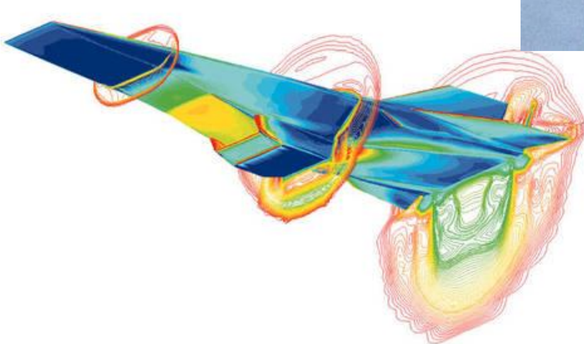
Subsonic flows: M<1.0

Transonic flows: M≈1.0

Supersonic flows: M>1.0

Hypersonic flows: M>5.0

Sonic boom



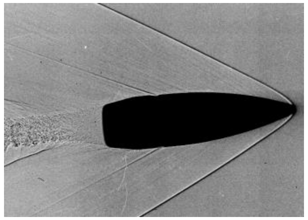
Dryden Flight Research Center ED97 43968-01
HYPER-X AT MACH 7: This computational fluid dynamic (CFD) image is of

Subsonic and Supersonic Flow

a. Stationary sound source

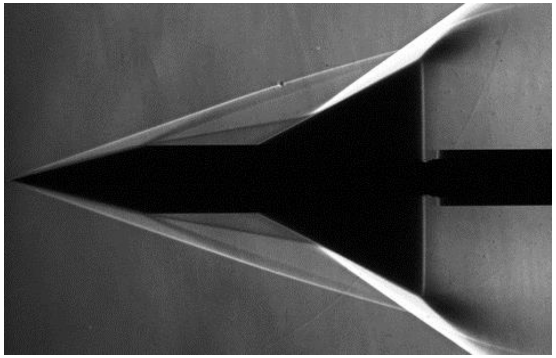
b. Source moving with $V_{source} < V_{sound}$

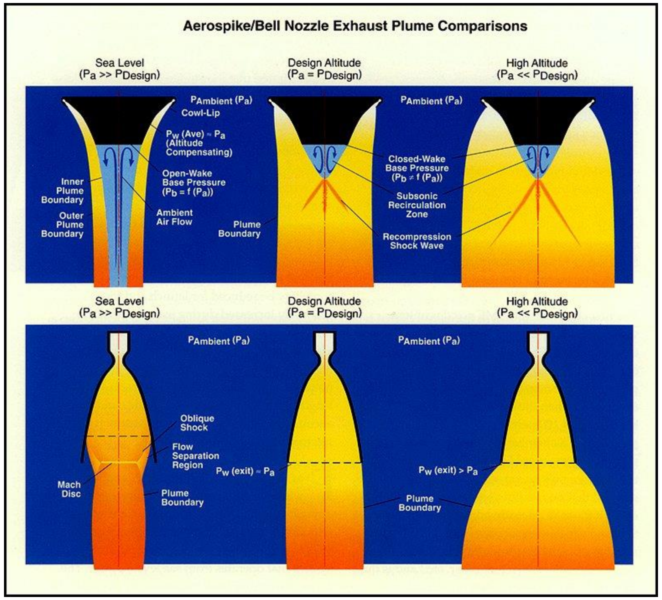
c. Source moving with $V_{source} = V_{sound}$
(Mach 1 - breaking the sound barrier)

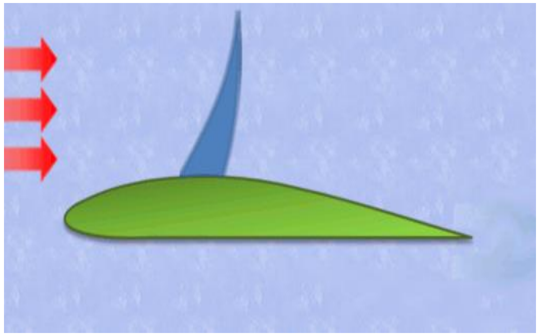


d. Source moving with $V_{source} > V_{sound}$
(Mach 1.4 - supersonic)

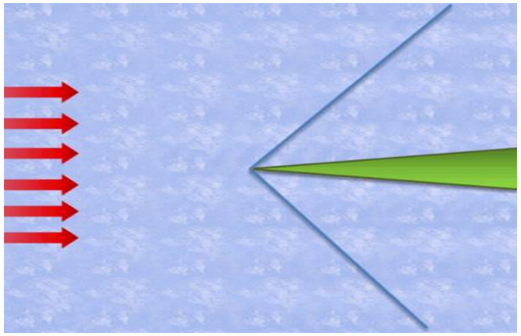
Shock Waves



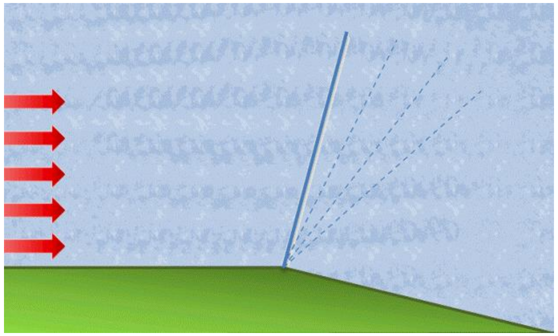




Normal Shock Wave
(The airstream slows to subsonic)



Oblique Shock Wave
(The airstream slows down, but remains supersonic)



Expansion Wave
(The airsteam accelerates, and the air behind the shock wave is higher supersonic)

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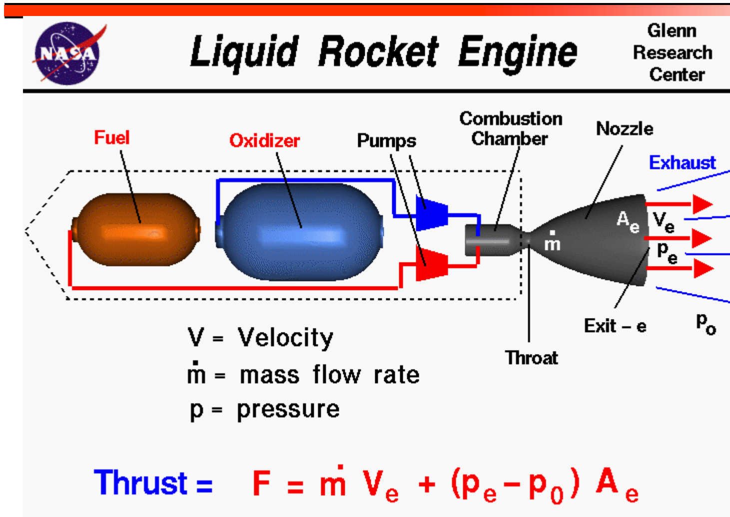
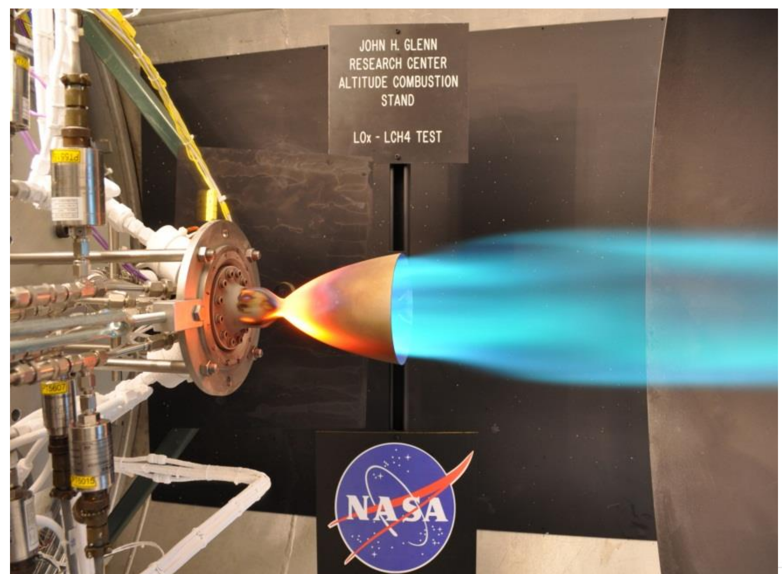
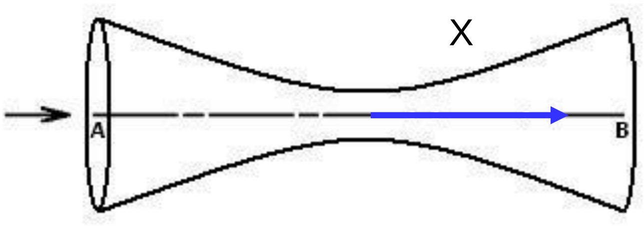
Review of Quasi-1D Nozzle Flow





Review of Quasi-1D Nozzle Flow

Assumptions:
- Steady-state
- Inviscid
- No body forces



Quasi-1D:
- Area is allowed to vary but flow variables are considered a function of x only

Mass conservation

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iint_S \rho \bar{U} \cdot \bar{n} dS = 0$$

Momentum conservation

$$\frac{\partial}{\partial t} \iiint_V \rho \bar{U} dV + \iint_S \rho \left(\bar{U} \cdot \bar{n} \right) \bar{U} dS = -\iint_S p dS + \iiint_V \rho \bar{f} dV + F_{viscous}$$

Energy conservation

$$\frac{\partial}{\partial t} \iiint_V \rho \left(e + \frac{U^2}{2} \right) dV + \iint_S \rho \left(e + \frac{U^2}{2} \right) \bar{U} \cdot \bar{n} dS = -\iint_S p \bar{U} \cdot \bar{n} dS + \iiint_V \rho \frac{\partial q}{\partial t} dV + \iiint_V \rho \left(\bar{f} \cdot \bar{U} \right) dV$$

Review of Quasi-1D Nozzle Flow

Assumptions:
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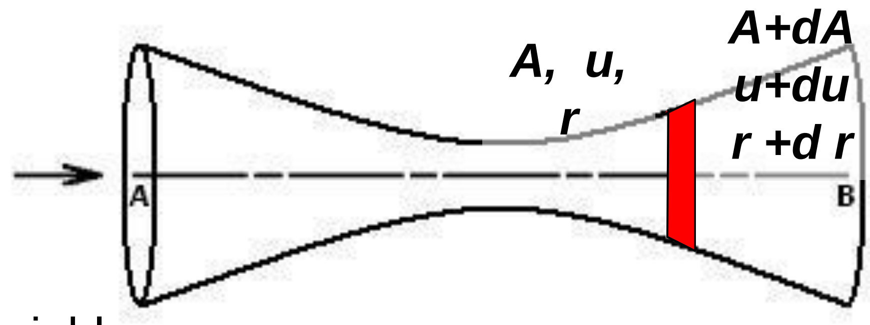
- Inviscid
- No body forces



Quasi-1D:
- Area is allowed to vary but flow variables are considered a function of x only

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iint_S \rho \bar{U} \cdot \bar{n} dS = 0$$

Mass conservation

$$-\rho u A + (u + du)(\rho + d\rho)(A + dA) = 0$$

$$-\cancel{\rho u A} + \cancel{\rho u A} + \rho u dA + \rho du A + d\rho u A + \text{higher order terms} = 0$$

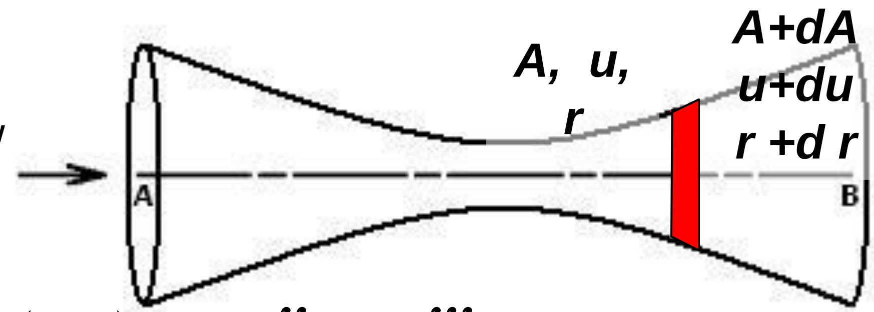
$$\rho A u = Const.$$

$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0$$

Review of Quasi-1D Nozzle Flow

Assumptions: Steady-state, Inviscid, No body forces

Quasi-1D: Area is allowed to vary but flow variables are considered a function of x only



Momentum conservation

$$\frac{\partial}{\partial t}\iiint_V \rho \bar{U} dV + \iint_S \rho(\bar{U}\cdot\bar{n})\bar{U} dS = -\iint_S p dS + \iiint_V \rho \bar{f} dV + F_{viscous}$$

$$-\rho u^2 A + (\rho + d\rho)(u+du)(u+du)(A+dA) = PA - (P+dP)(A+dA) + 2\left(\frac{PdA}{2}\right)$$

$$-\cancel{\rho u^2 A} + \cancel{\rho u^2 A} + \rho u^2 dA + u^2 A d\rho + \rho u A du + \rho u A du = \cancel{PA} - \cancel{PA} - \cancel{PdA} - AdP + \cancel{PdA}$$

$$u\left(\cancel{\rho u dA + uAd\rho + \rho A du}\right) + \rho u A du = -AdP$$

$$dP = -\rho u du \quad \Rightarrow \quad \frac{dP}{\rho} = \frac{dP}{d\rho}\frac{d\rho}{\rho} = -udu \quad \left| a^2 = \left.\frac{dP}{d\rho}\right)_s \right. \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{u}{a^2}du$$

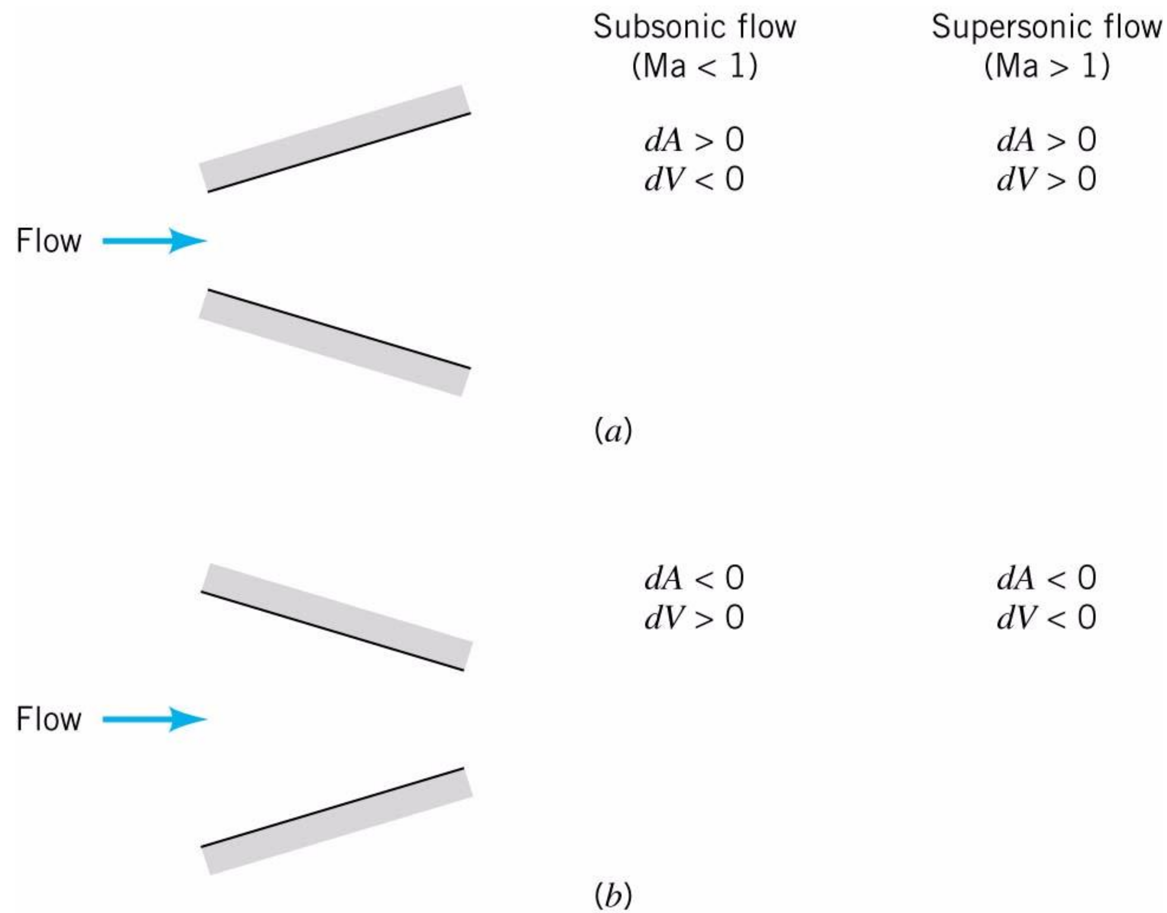
Since:

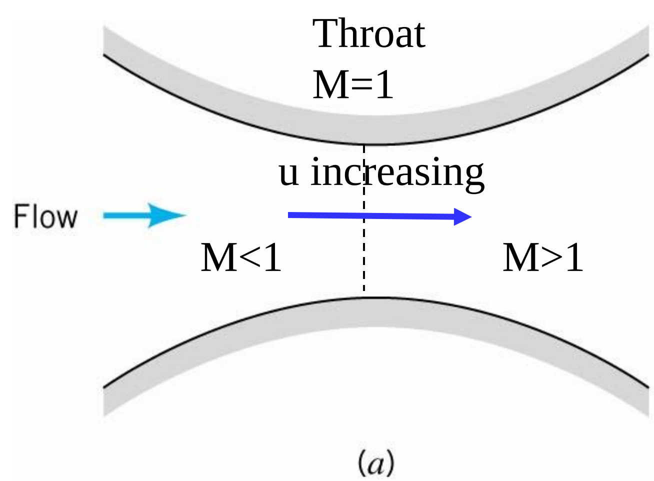
$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0 \quad \Rightarrow \quad \frac{dA}{A} + \frac{du}{u} - \frac{u}{a^2}du = 0$$

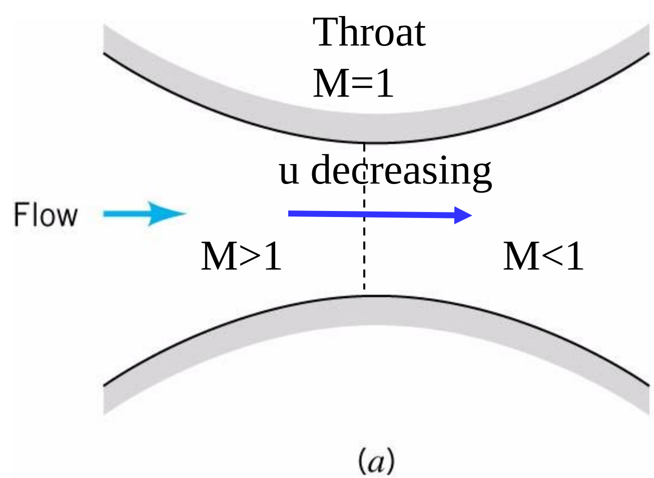
$$\Rightarrow \quad \frac{dA}{A} + \frac{du}{u}\left(1 - \frac{u^2}{a^2}\right) = \frac{dA}{A} + \frac{du}{u}(1 - M^2) = 0 \quad \Rightarrow \quad \frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

Review of Quasi-1D Nozzle Flow

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$







Review of Quasi-1D Nozzle Flow

$$Isentropic\ \ relations\ \ (\text{Thermodynamics}):$$

$$\frac{P_2}{P_1} = (\frac{\rho_2}{\rho_1})^{\gamma} = (\frac{T_2}{T_1})^{\frac{\gamma}{\gamma-1}}$$

$$Energy\quad Equation:$$

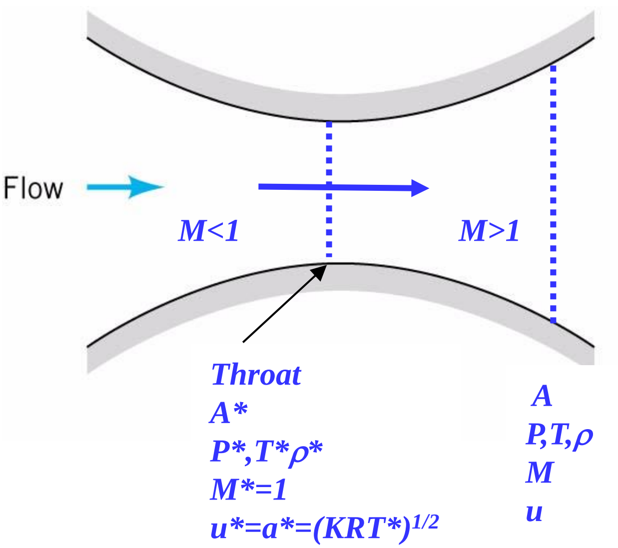
$$C_P T_0 = C_P T + \frac{V^2}{2} \Rightarrow \frac{T_0}{T} = 1 + \frac{V^2}{2C_P T} = 1 + \frac{\gamma-1}{2}\frac{V^2}{\gamma RT}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2$$

$$\frac{P_0}{P} = (1 + \frac{\gamma-1}{2} M^2)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = (1 + \frac{\gamma-1}{2} M^2)^{\frac{1}{\gamma-1}}$$

Review of Quasi-1D Nozzle Flow



$$\rho^* u^* A^* = \rho u A$$

$$at\ A^*\ \text{section}:\quad \mathrm{M} = \frac{u^*}{a^*} = 1 \quad \Rightarrow u^* = a^*$$

isentropic relation

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2$$

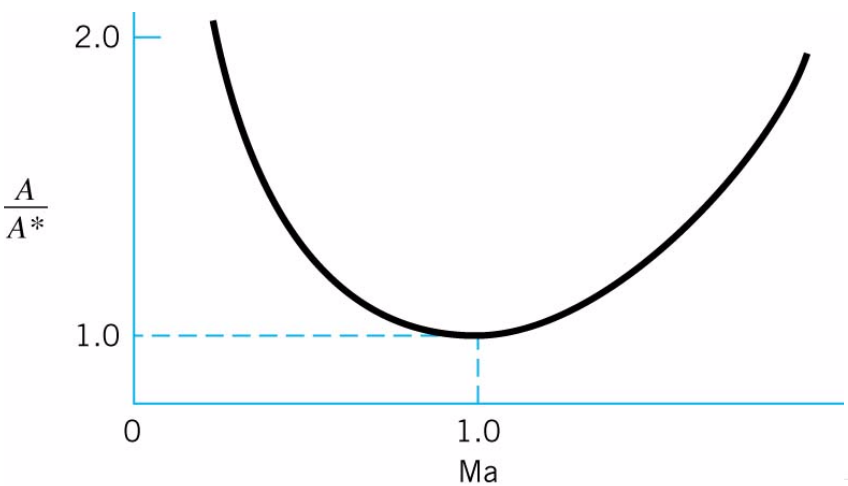
$$\frac{P_0}{P} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

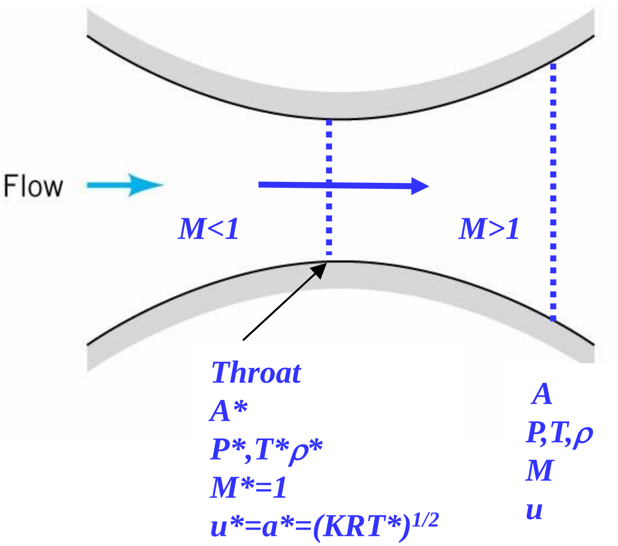
$$\Rightarrow \left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho}\right)^2 \left(\frac{a^*}{u}\right)^2 = \left(\frac{\rho^*}{\rho_0}\right)^2 \left(\frac{\rho_0}{\rho}\right)^2 \left(\frac{a}{u}\right)^2 \left(\frac{a^*}{a}\right)^2$$

$$= \frac{1}{M^2}\left[\frac{2}{\gamma-1}\left(1 + \frac{\gamma-1}{2} M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma-1}\left(1 + \frac{\gamma-1}{2} M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$



Review of Quasi-1D Nozzle Flow

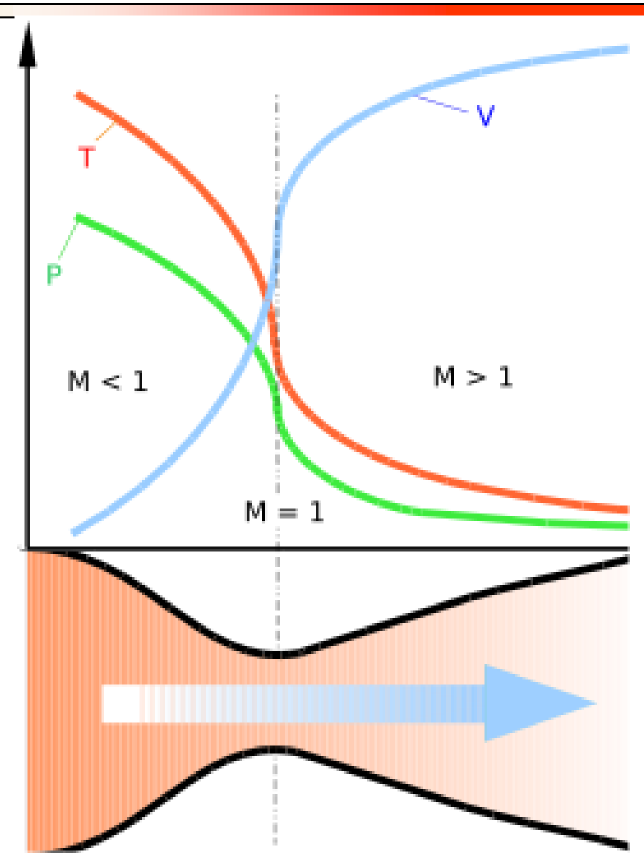


$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma-1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

$$\frac{T_0}{T} = 1+\frac{\gamma-1}{2}M^2$$

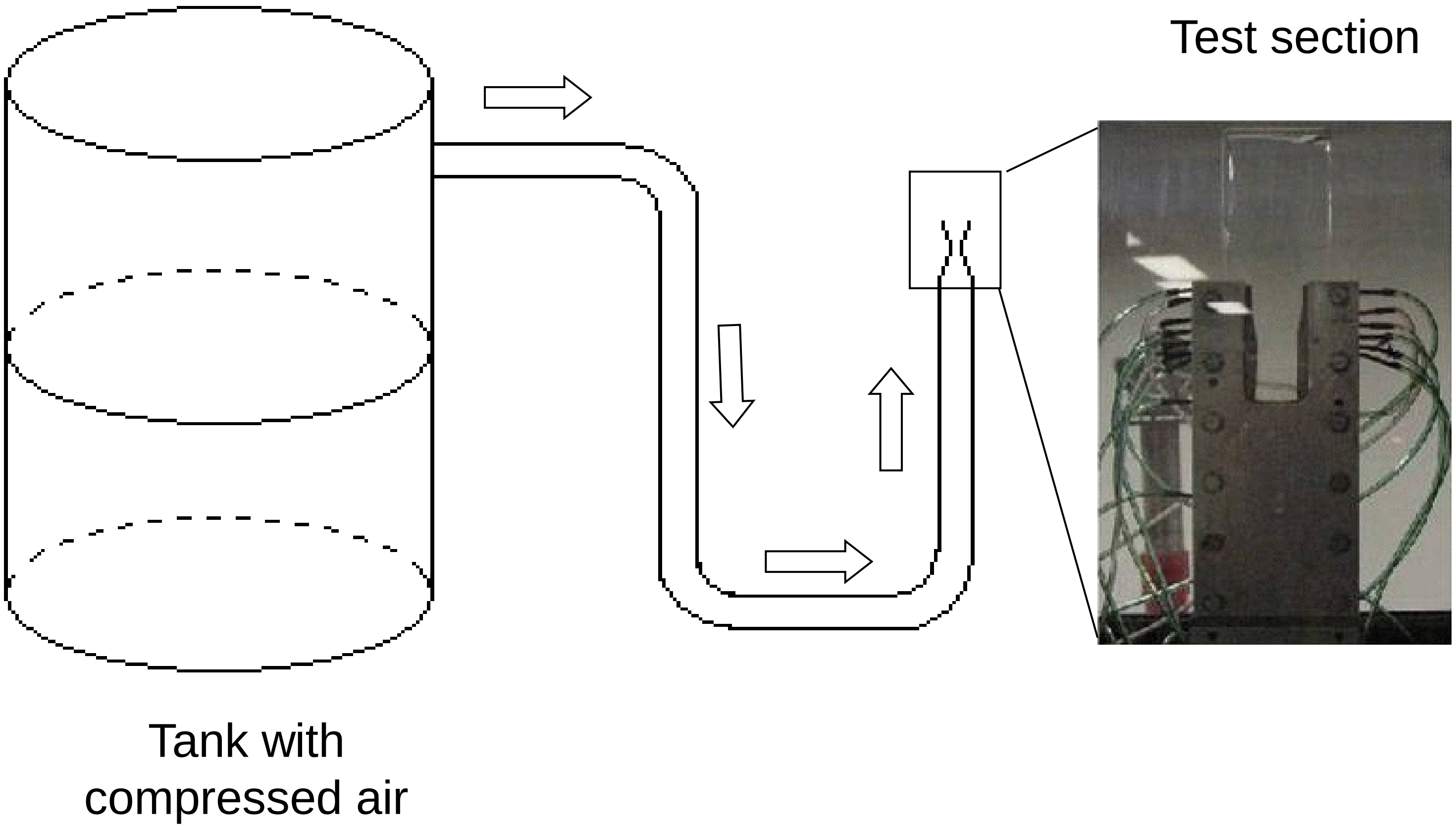
$$\frac{P_0}{P} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

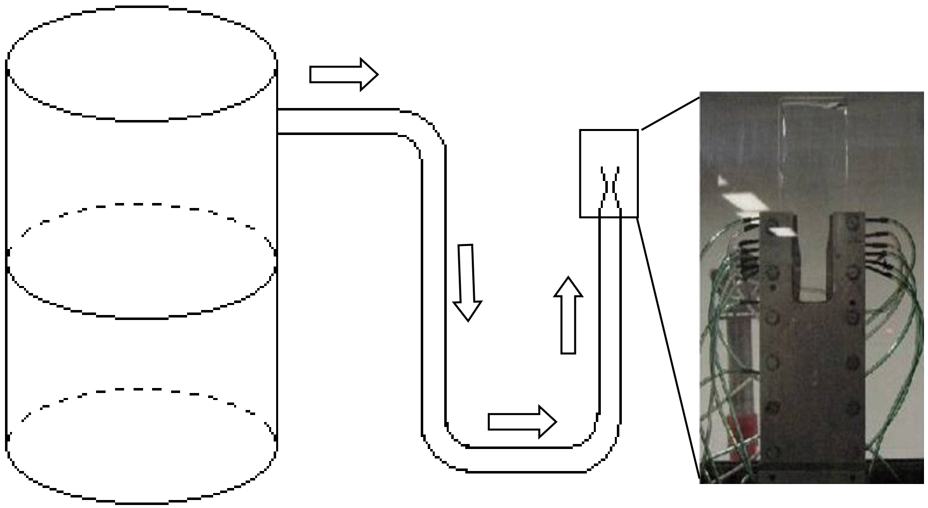


de Laval nozzle
(or convergent-divergent nozzle, CD nozzle) is a tube that is pinched in the middle, making an hourglass-shape. It is used as a means of accelerating the flow of a gas passing through it to a supersonic speed

Test Section

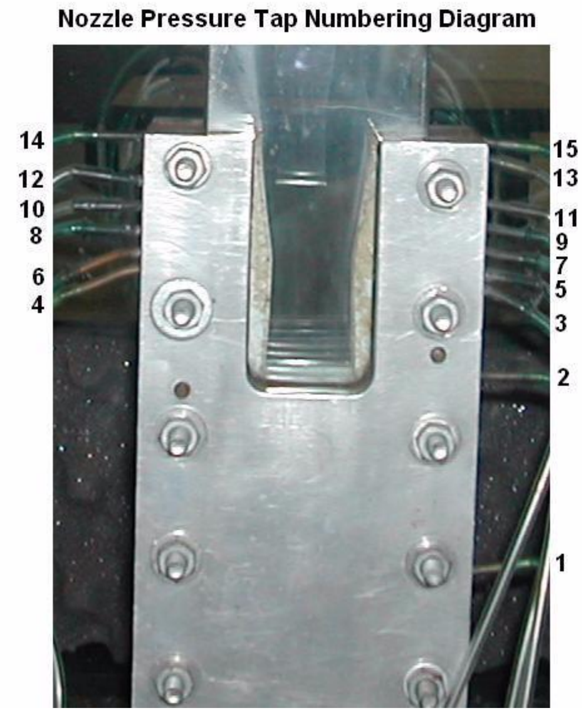


AerE344 Lab: Pressure Measurements in a de Laval Nozzle



Test section

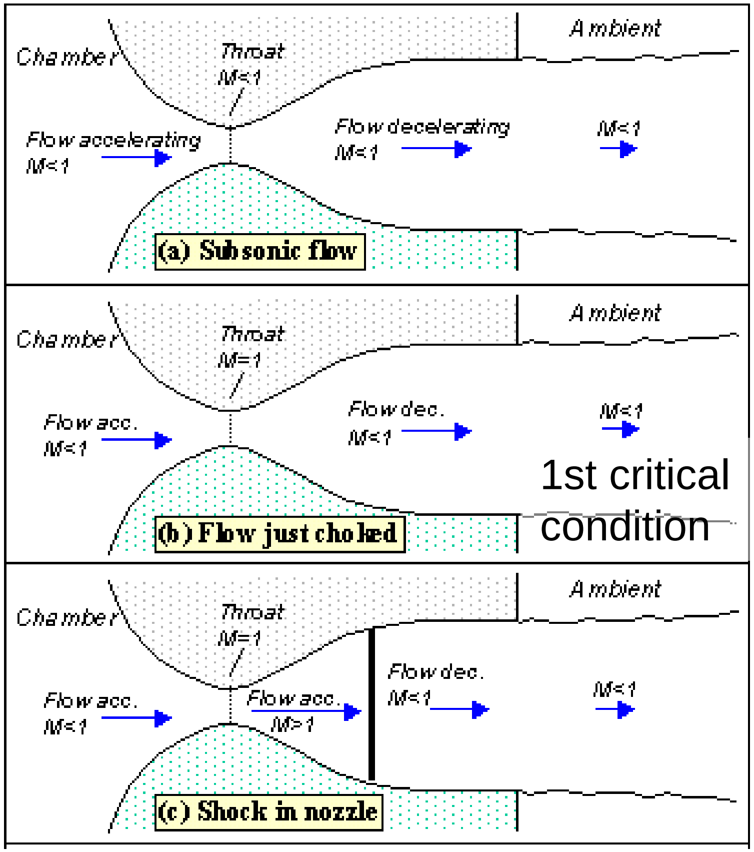
Tank with compressed air



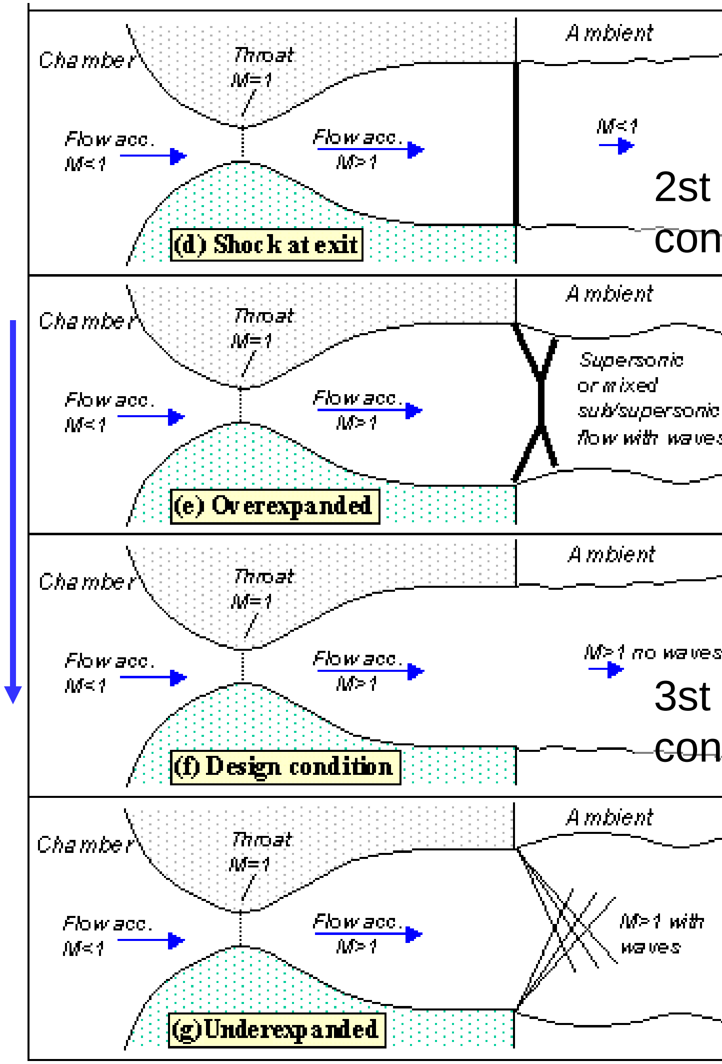
Tap No.	Distance downstream of throat (inches)	Area (Sq. inches)
1	-4.00	0.800
2	-1.50	0.529
3	-0.30	0.480
4	-0.18	0.478
5	0.00	0.476
6	0.15	0.497
7	0.30	0.518
8	0.45	0.539
9	0.60	0.560
10	0.75	0.581
11	0.90	0.599
12	1.05	0.616
13	1.20	0.627
14	1.35	0.632
15	1.45	0.634

1st, 2nd, and 3rd critical conditions

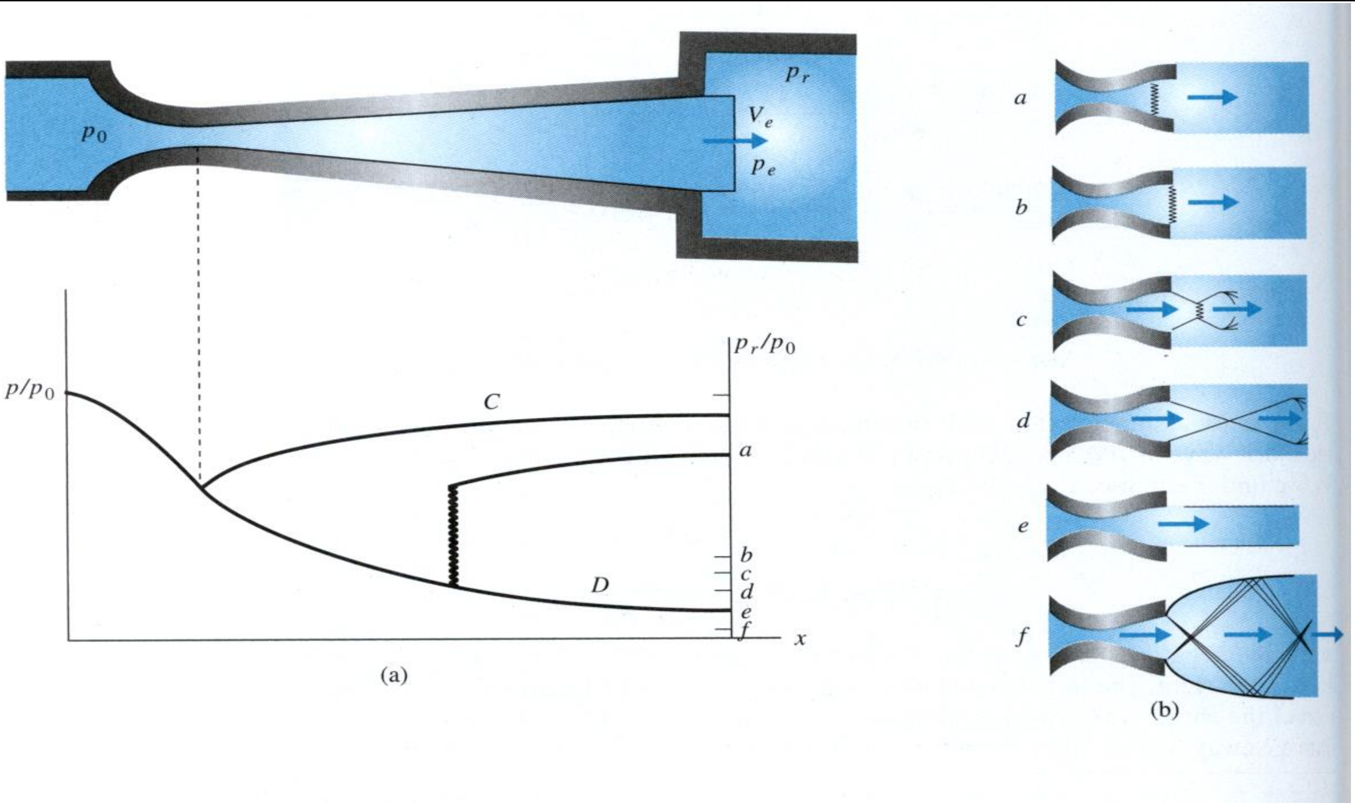
P_0 increasing



P_0 increasing

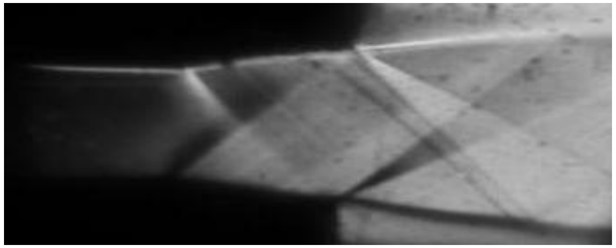


AerE344 Lab: Pressure Measurements in a de Laval Nozzle

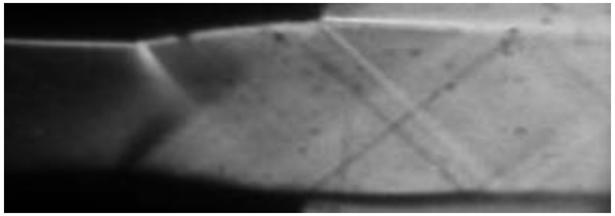


1. Under-expanded flow
2. 3rd critical
3. Over-expanded flow with oblique shocks
4. 2nd critical
5. Normal shock existing inside the nozzle
6. 1st critical

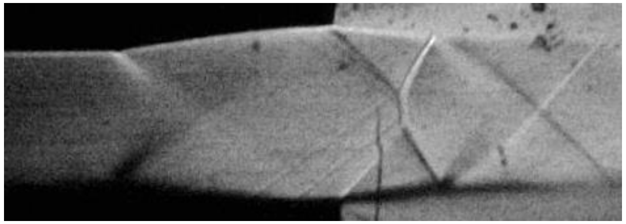
1st, 2nd, and 3rd critical conditions



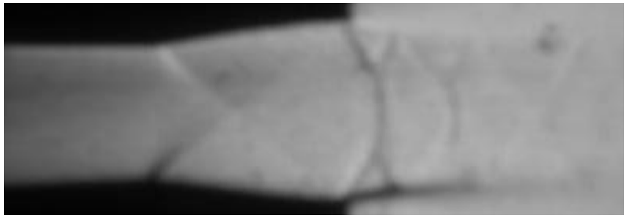
Under-expanded flow



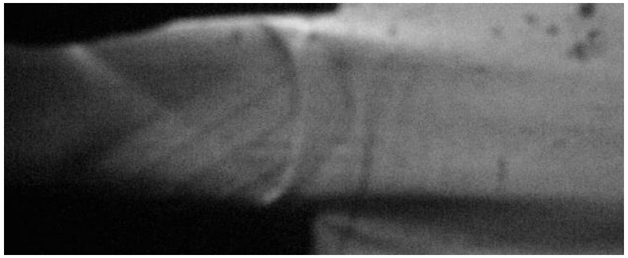
Flow close to 3rd critical



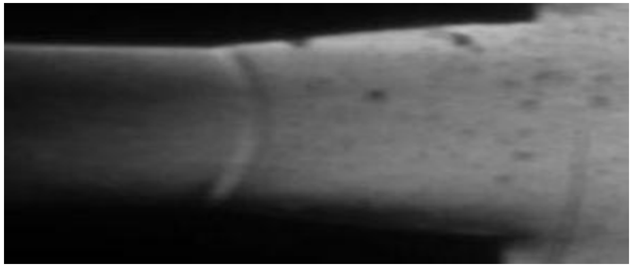
Over-expanded flow



2nd critical – shock is at nozzle exit

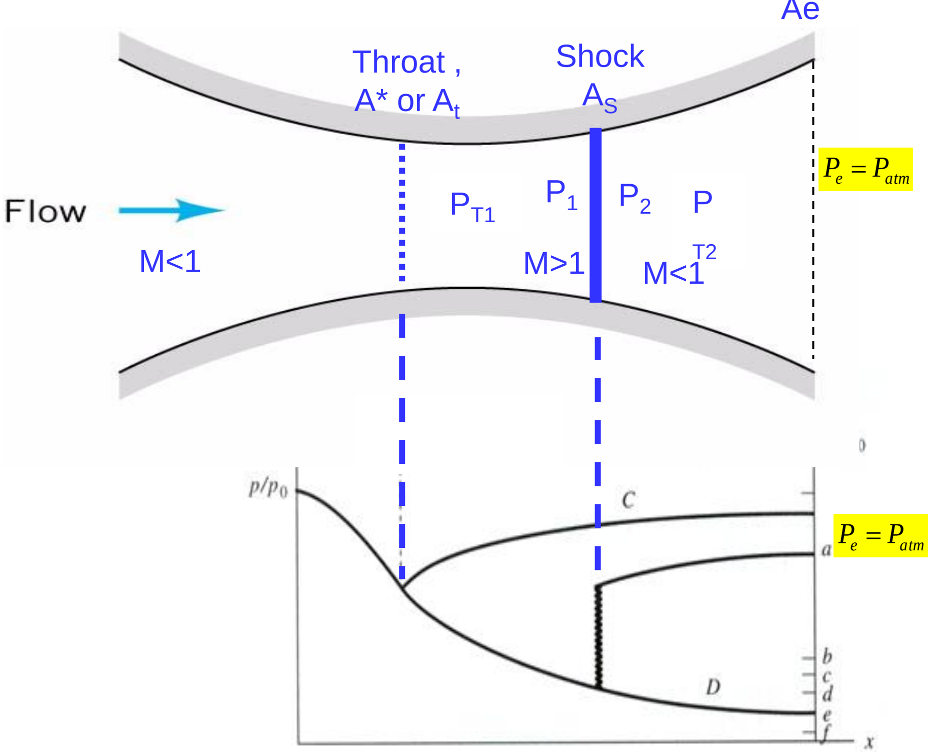


Over-expanded flow with shock between nozzle exit and throat



1st critical – shock is almost at the nozzle throat.

Pressure Distribution Prediction within a De Laval Nozzle by using Numerical Approach



- Using the area ratio, the Mach number at any point up to the shock can be determined:

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

- After finding Mach number at front of shock, calculate Mach number after shock using:

$$M_2^{\,2} = \frac{1+\frac{\gamma-1}{2}M_1^{\,2}}{\gamma M_1^{\,2}-\frac{\gamma-1}{2}}$$

- Then, calculate the A_2*

$$\left(A_2^{\,*}\right)^2 = M_2^{\,2}A_s^{\,2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)\right]^{-\frac{\gamma+1}{\gamma-1}}$$

which allows us calculate the remaining Mach number distribution

$$\left(\frac{A}{A_2^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

d. To calculate Mach number given the Mach-Area relation, can use Newton iteration to find M

$$F = \left(\frac{A}{A^*}\right)^2 = M^2\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} \quad (2.8)$$

$$F' = \frac{dF}{dM} = \frac{2}{M^3}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} - \frac{2}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{2}{\gamma-1}} \quad (2.9)$$

$$M^{n+1} = M^n - \frac{F}{F'} \quad (2.10)$$

Pressure Distribution Prediction within a De Laval Nozzle by using Numerical Approach

2. Find pressure distribution
   a. Pressure at exit is same as atmospheric pressure for shock inside nozzle ($P_e = P_{atm}$). For shock after lip of nozzle, total pressure is constant throughout the interior of the nozzle ($P_{t2} = P_{t1}$).

   b. Find total pressure behind the shock:

$$P_{t2} = \frac{P_{t2}}{P_e} P_e \text{ where } \frac{P_{t2}}{P_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}} \qquad (2.4)$$

   c. Any pressure behind the shock is therefore:

$$P = P_{t2}\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{-\gamma}{\gamma - 1}} \qquad (2.5)$$

   d. Calculate P_{t1} ahead of shock:

$$P_{t1} = \frac{P_{t1}}{P_1} \frac{P_1}{P_2} \frac{P_2}{P_{t2}} P_{t2} \qquad (2.6)$$

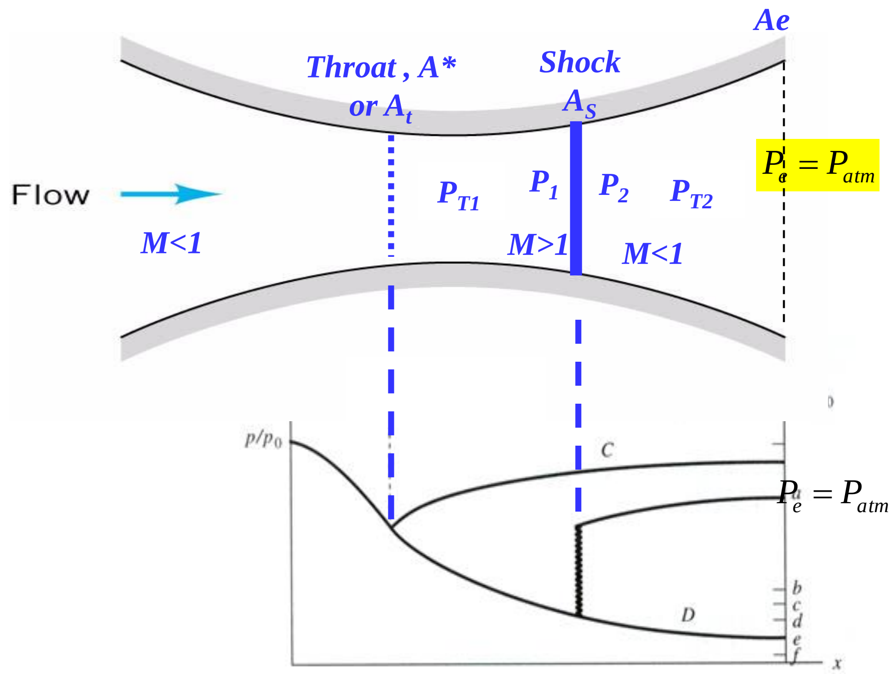
   where you can use Total-Static relation for 1st and 3rd ratios, and for the middle ratio:

$$\frac{P_1}{P_2} = \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}$$

$$or \qquad (2.7)$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$

   e. Now that you have the total pressure upstream of the shock, as well as the Mach number calculated earlier you can calculate the pressure upstream of the shock.

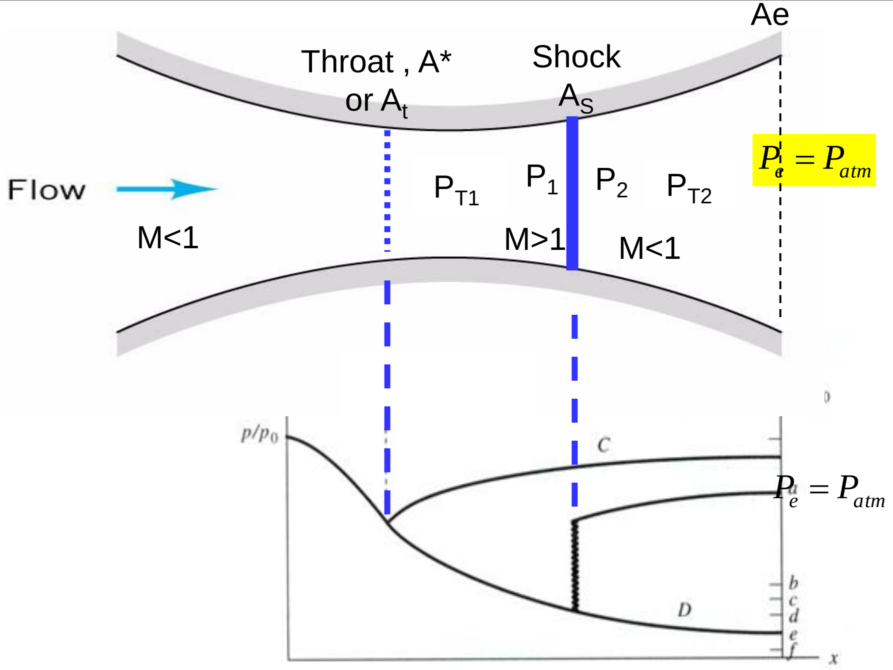


a. For 3rd Critical
   1. $P_1 = P_2 = P_e$
   2. $M_1 = M_2 = M_e$ (supersonic)

b. For 1st Critical
   1. Same as 3rd critical, but M_e is subsonic

c. For 2nd Critical
   1. $M_2 = M_e$
   2. $P_2 = P_e$

Pressure Distribution Prediction within a De Laval Nozzle by using Numerical Approach



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15	1.45	0.634

- Method #1, by using equations:

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

- Method #2: by using Isentropic Flow properties table (Appendix-A of Anderson's textbook)

- If the shockwave is located at position of tap#12:

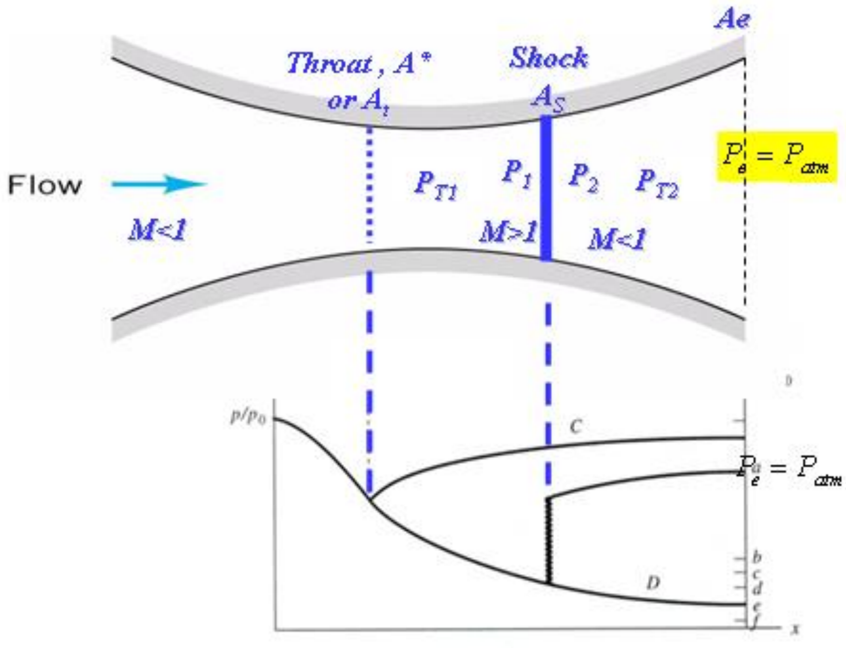
Tap No.	A/A*	Mach #	P/P_t	P	P_g
1	1.681	0.37	0.9098		
2	1.111	0.67	0.7401		
3	1.008	0.97	0.5469		
5	1.000	1.00	0.5283		
7	1.088	1.35	0.3370		
9	1.176	1.50	0.2724		
11	1.258	1.61	0.2318		
pre-shock	1.294	1.64	0.2217		
post-shock					
13					
15					

Pressure Distribution Prediction within a De Laval Nozzle by using Table Method

- By using the normal shock tables with M1 = 1.64 we find that M2 = 0.686. (Appendix-B of Anderson's textbook)
- Next, we find the sonic reference area behind the shock using the area-Mach relation. i.e., M2=0.686 (Appendix-A of Anderson's textbook)
- Find sonic reference behind the shock using the area-Mach relationship:

$$\left(A_2^*\right)^2 = A_S^{\,2} M_2^{\,2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)\right]^{-\frac{\gamma+1}{\gamma-1}}$$

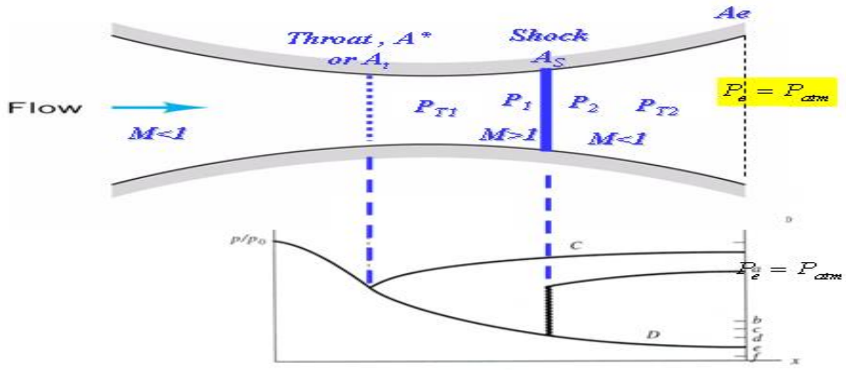
i.e., A2*=0.557sq. Inches



- If the shockwave is located at position of tab#12:

Tap No.	A/A*	Mach #	P/P_t	P	P_g
1	1.681	0.37	0.9098		
2	1.111	0.67	0.7401		
3	1.008	0.97	0.5469		
5	1.000	1.00	0.5283		
7	1.088	1.35	0.3370		
9	1.176	1.50	0.2724		
11	1.258	1.61	0.2318		
pre-shock	1.294	1.64	0.2217		
post-shock	1.105	0.69	0.7274		
13					
15					

Pressure Distribution Prediction within a De Laval Nozzle by using Table Method



- With the exit pressure to be sea-level standard pressure. We now calculate the total pressure behind the shock using this value of exit pressure and the pressure ratio at the exit:

$$P_{t2} = \frac{P_t}{P} P = \left(\frac{1}{0.7528}\right) 14.7 = 19.53$$

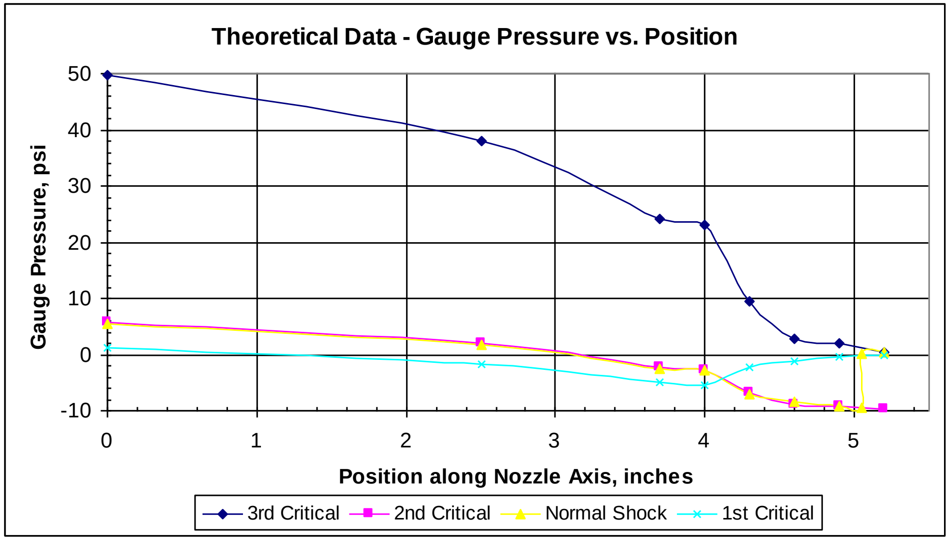
Tap No.	A/A*	Mach #	P/P_t	P	P_g
1	1.681	0.37	0.9098		
2	1.111	0.67	0.7401		
3	1.008	0.97	0.5469		
5	1.000	1.00	0.5283		
7	1.088	1.35	0.3370		
9	1.176	1.50	0.2724		
11	1.258	1.61	0.2318		
pre-shock	1.294	1.64	0.2217		
post-shock	1.105	0.69	0.7274	14.21	
13	1.125	0.66	0.7465	14.58	
15	1.137	0.65	0.7528	14.7	

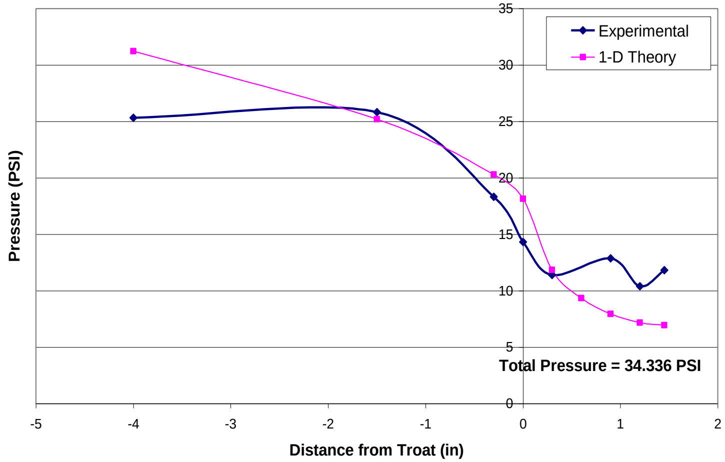
- Our last major task is to find the total pressure ahead of the shock, P_{t1}

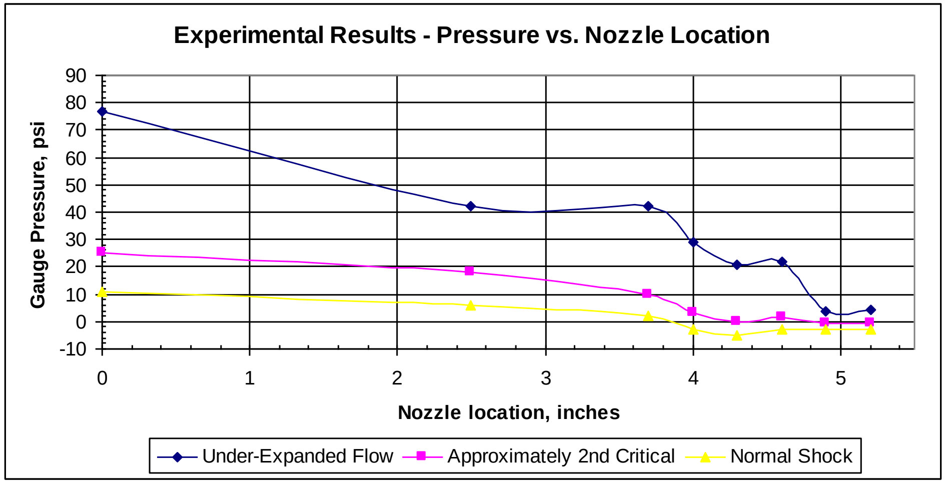
$$P_{t1} = \frac{P_{t1}}{P_1}\frac{P_1}{P_2}\frac{P_2}{P_{t2}} P_{t2}$$

Tap No.	A/A*	Mach #	P/P_t	P	P_g
1	1.681	0.37	0.9098	19.6	4.9
2	1.111	0.67	0.7401	16	1.3
3	1.008	0.97	0.5469	11.8	-2.9
5	1.000	1.00	0.5283	11.4	-3.3
7	1.088	1.35	0.3370	7.27	-7.43
9	1.176	1.50	0.2724	5.88	-8.82
11	1.258	1.61	0.2318	5	-9.7
pre-shock	1.294	1.64	0.2217	4.78	-9.92
post-shock	1.105	0.69	0.7274	14.21	-0.49
13	1.125	0.66	0.7465	14.58	-0.12
15	1.137	0.65	0.7528	14.7	0

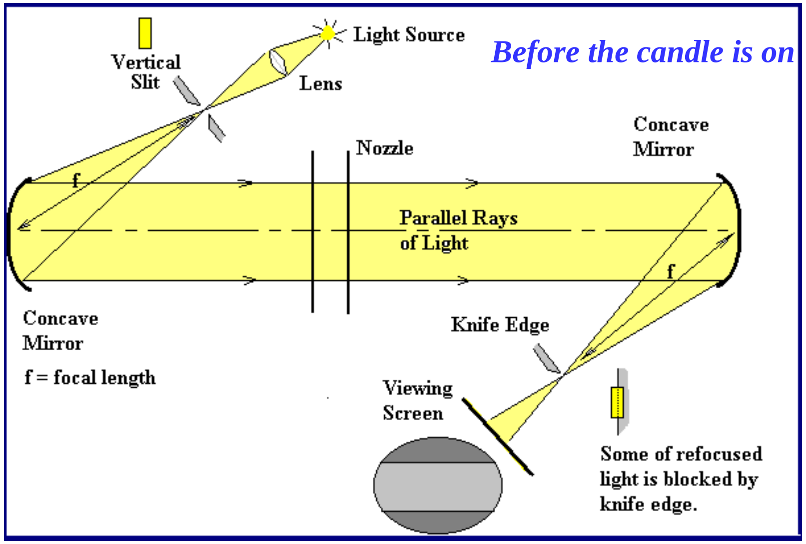
Examples of the previous lab reports

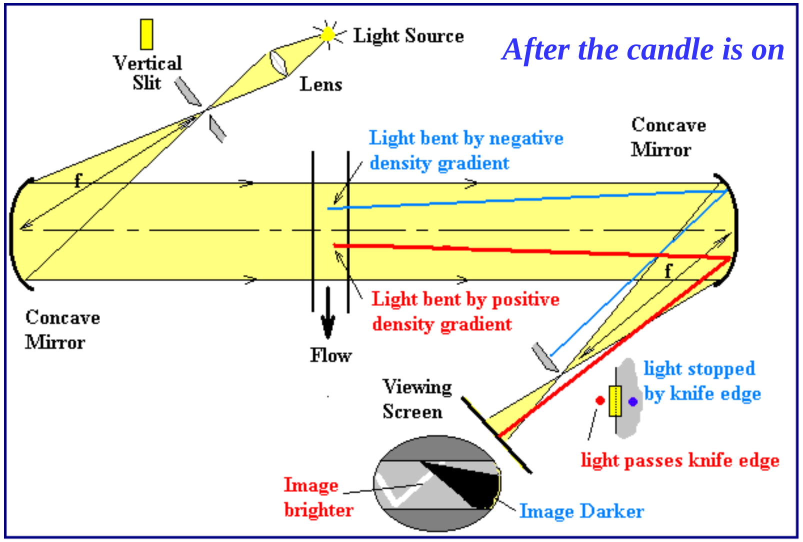


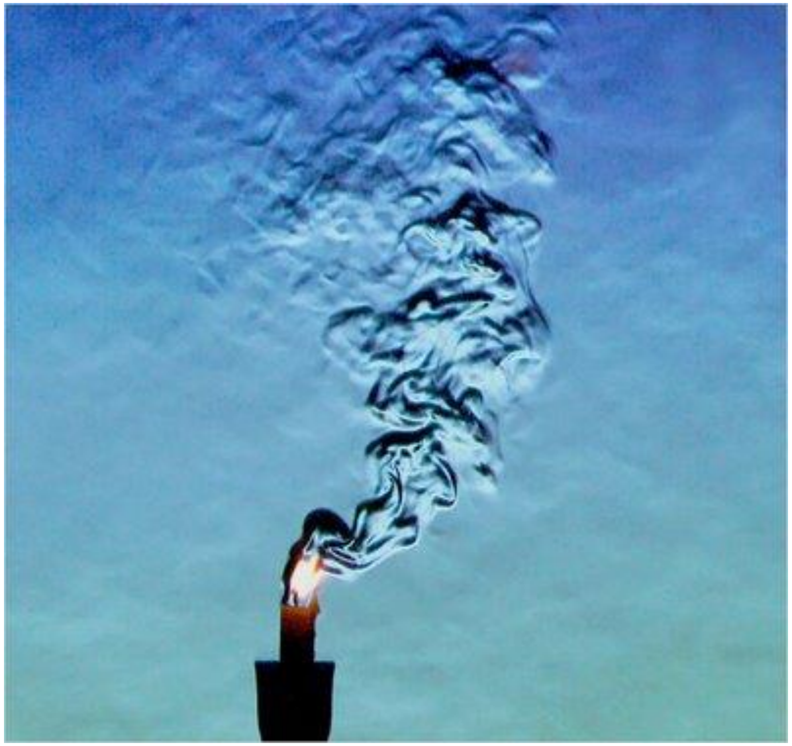




AerE344 Lab#10: Set up Schlieren and Shadowgraph Systems to Visualize a Thermal Plume Flow of a Burning Candle







Schlieren image of a the thermal plume of a burning candle

AerE344 Lab#10: Set up Schlieren and Shadowgraph Systems to Visualize a Thermal Plume Flow of a Burning Candle

- Set up Schlieren and Shadowgraph Systems to visualize a thermal plume flow .
- Sign-in sheet signature.
- No lab report.